

# FUZZY COSETS AND QUOTIENT FUZZY AG-SUBGROUPS

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**ABSTRACT.** In this paper we extend the concept of fuzzy AG-subgroups. We introduce some results in *normal fuzzy AG-subgroups*. We define *fuzzy cosets* and *quotient fuzzy AG-subgroups*, and prove that the sets of their collection form an AG-subgroup and fuzzy AG-subgroup respectively. We also introduce the fuzzy Lagrange's Theorem of AG-subgroup. It is known that the condition  $\mu(xy) = \mu(yx)$  holds for all  $x, y$  in fuzzy subgroups if  $\mu$  is normal, but in fuzzy AG-subgroup we show that it holds without normality.

## 1. INTRODUCTION

The concept of fuzzy sets along with various operations has been introduced by Lofti A. Zadeh in 1965 [1]. Due to the diverse applications ranging from engineering, computer science and social behavior studies, the researchers have taken keen interest in the subject in its related fields. The study of fuzzy algebraic structures was started by introducing the concept of fuzzy subgroups by A. Rosenfeld [2]. He formulated the concept of fuzzy subgroup and extend the main idea of group theory to develop the theory of fuzzy groups. Anthony and Sherwood further redefined fuzzy groups [3]. Many other papers on fuzzy subgroups have also appeared which generalize various concepts of group theory such as normal subgroups, quotient groups and cosets [4, 5, 6].

In the forty years history of AG-groupoids, though it was explored slowly, yet in the last couple of years abundant research was carried out in this area which attracted the attention of many new researchers. An AG-groupoid is a generalization of commutative semigroup. It is

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a nonassociative groupoid in general, in which the left invertive law  $(ab)c = (cb)a$  holds. In general, an AG-group is a nonassociative structure in which commutativity and associativity imply each other, and thus it becomes abelian group if any one of them is allowed. An AG-groupoid  $(G, \cdot)$  is called an AG-group or left almost group (LA-group), if there exists a unique left identity  $e \in G$  (that is  $ea = a$  for all  $a \in G$ ), and for all  $a \in G$  there exists  $a^{-1} \in G$  such that  $a^{-1}a = aa^{-1} = e$ . M. Kamran extended the notion of AG-groupoid to an AG-group and defined cosets of an AG-subgroup  $H$  of an AG-group  $G$  and proved that quotient  $G/H$  is defined for every AG-subgroup  $H$ . He also proved that Lagrange's Theorem holds for AG-group [7]. The third author of this article has discussed various basic properties of AG-groups and explored new results such as: complexes and cosets decomposition, conjugacy relations in AG-groups, normality, normalizers and many more [8, 13]. For the first time in 2003, Q. Mushtaq and M. Khan introduced ideals in AG-groupoid and fuzzified these concepts [9]. This attracted the attention of various other researchers to the field of AG-groupoids and AG-groups, as a result since then we can see lots of papers in this area. It is also worth mentioning that various new classes of AG-groupoids have been recently introduced [12, 14, 15, 16, 17, 18] and some are just have been arXived [19, 20] and their fuzzification is suggested as an interesting future work.

In this paper we extend the concepts of normal fuzzy AG-subgroup [10, 11]. We further define fuzzy cosets, quotient AG-subgroups and quotient fuzzy AG-subgroups, which will provide new direction to the researchers in this area. We also introduce a fuzzy version of the famous Lagrange's Theorem for finite AG-groups.

## 2. PRELIMINARIES

In this section we list some basic definitions that will frequently be used in the subsequent sections of the paper.

A *fuzzy subset*  $\mu$  is a mapping  $\mu : X \rightarrow [0, 1]$ . The set of all fuzzy subsets of  $X$  is called the *fuzzy power set* of  $X$  and is denoted by  $FP(X)$ . Let  $\mu \in FP(X)$ , then the the image of  $\mu$  is a set  $\{\mu(x) : x \in X\}$  and is denoted by  $\mu(X)$  or  $Im(\mu)$ .

In the rest of this paper  $G$  will denote an AG-group otherwise stated and  $e$  will denote the left identity of  $G$ .

**Definition 1.** [10] Let  $\mu \in FP(G)$ , then  $\mu$  is called a fuzzy AG-subgroup of  $G$  if for all  $x, y \in G$ ;

- (i)  $\mu(xy) \geq \mu(x) \wedge \mu(y)$ ;
- (ii)  $\mu(x^{-1}) \geq \mu(x)$ .

The set of all fuzzy AG-subgroups of  $G$  is denoted by  $F(G)$ .

If  $\mu \in F(G)$ , then

$$(2.1) \quad \mu_* = \{x \in G \mid \mu(x) = \mu(e)\}.$$

**Lemma 1.** [10] Let  $\mu$  be any fuzzy AG-subgroup of  $G$  i.e  $\mu \in F(G)$ , then for all  $x \in G$ ,

- (i)  $\mu(e) \geq \mu(x)$ ;
- (ii)  $\mu(x) = \mu(x^{-1})$ .

**Definition 2.** [10] Let  $\mu \in F(G)$ . Then  $\mu$  is called a normal fuzzy AG-subgroup of  $G$  if

$$\mu(xy \cdot x^{-1}) = \mu(y) \quad \forall x, y \in G.$$

The set of all normal fuzzy AG-subgroups of  $G$  is denoted by  $NF(G)$ .

**Example 1.** Consider an AG-group of order 4:

$\cdot$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

define fuzzy subset  $\mu$  by  $\mu(0) = t_0$  and  $\mu(x) = t_1$  otherwise; where  $t_0, t_1 \in [0, 1]$  and  $t_0 > t_1$ . Then  $\mu$  is fuzzy AG-subgroup. However,  $\mu$  is not normal fuzzy AG-subgroup. Here  $\mu_* = \{0\}$  which is normal fuzzy AG-subgroup.

**Example 2.** Consider an AG-group  $G = \langle a, b : a^2 = b^2 = (ab)^2 = e \rangle$ , define fuzzy subset  $\mu : G \rightarrow [0, 1]$  by  $\mu(e) = t_0$ ,  $\mu(a) = t_1$  and  $\mu(x) = t_2$  otherwise; where  $t_0, t_1, t_2 \in [0, 1]$  and  $t_0 > t_1 > t_2$ . Then  $\mu$  is normal fuzzy AG-subgroup of  $G$ .

### 3. MAIN RESULTS

**Proposition 1.** Let  $\mu \in F(G)$ . Then  $\mu(xy) = \mu(yx) \forall x, y \in G$ .

*Proof.* Let  $\mu \in F(G)$ , then

$$\begin{aligned} \mu(xy) &= \mu(ex \cdot y) = \mu(yx \cdot e) && [\text{using left invertive law}] \\ &\geq \mu(yx) \wedge \mu(e) = \mu(yx) && [\mu(e) \geq \mu(yx) \forall x, y \in G] \\ \Rightarrow \mu(xy) &\geq \mu(yx) \quad \forall x, y \in G. \end{aligned}$$

Similarly, we can show that  $\mu(yx) \geq \mu(xy) \forall x, y \in G$ . Thus  $\mu(xy) \geq \mu(yx) \geq \mu(xy) \forall x, y \in G$ . Hence  $\mu(yx) = \mu(xy)$ . ■

**Lemma 2.** [10, Lemma 17] Let  $\mu$  be a fuzzy AG-subgroup of  $G$ . Let  $x \in G$  then  $\mu(xy) = \mu(y) \forall y \in G$  if and only if  $\mu(x) = \mu(e)$ .

**Theorem 1.** Let  $f$  be a homomorphism on AG-group  $G$  and  $\mu$  is any normal fuzzy AG-subgroup of  $f(G)$ . Then  $\mu \circ f \in NF(G)$ .

*Proof.* First we show that  $\mu \circ f \in F(G)$ . Since

$$\begin{aligned} \mu \circ f(xy) &= \mu(f(xy)) \\ &= \mu(f(x) \cdot f(y)) \\ &\geq \mu(f(x)) \wedge \mu(f(y)) \\ &= \mu \circ f(x) \wedge \mu \circ f(y) \end{aligned}$$

$\forall x, y \in G$ , and

$$\begin{aligned} \mu \circ f(x^{-1}) &= \mu(f(x^{-1})) \\ &= \mu((f(x))^{-1}) \\ &= \mu(f(x)) \\ &= \mu \circ f(x). \end{aligned}$$

$\forall x \in G$ . Hence  $\mu \circ f \in F(G)$ .

Next we show that  $\mu \circ f \in NF(G)$ , since

$$\begin{aligned}
 \mu \circ f(xy \cdot x^{-1}) &= \mu(f(xy \cdot x^{-1})) \\
 &= \mu(f(xy) \cdot f(x^{-1})) \\
 &= \mu(\{f(x)f(y)\} \cdot (f(x))^{-1}) \\
 &= \mu(f(y)) [\mu \in NF(f(G))] \\
 &= \mu \circ f(y).
 \end{aligned}$$

$\forall x, y \in G$ . Hence  $\mu \circ f \in NF(G)$ . ■

**Definition 3.** Let  $\mu \in F(G)$ , for any  $x \in G$  define a mapping

$$\mu_x : G \rightarrow [0, 1], \text{ by}$$

$$(3.1) \quad \mu_x(g) = \mu(gx^{-1}) \quad \forall g \in G.$$

Then  $\mu_x$  is called fuzzy coset of  $G$  determined by  $x$  and  $\mu$ , and the collection of all fuzzy cosets of  $\mu$  is represented by  $\mathcal{F}$ .

In AG-groups we can define quotient AG-group by any AG-subgroup without normality. Therefore, make use of this we can define quotient AG-groups or factor AG-group as follows:

**Theorem 2.** Let  $\mu \in NF(G)$  and  $G/\mu = \{\mu_x : x \in G\}$ . Then  $G/\mu$  form an AG-group under the usual composition of mappings define by  $\mu_x \circ \mu_y = \mu_{xy} \quad \forall x, y \in G$ .

*Proof.* First we show that the composition of cosets is well defined. Let  $x, y, x_\circ, y_\circ \in G$  such that  $\mu_x = \mu_{x_\circ}$  and  $\mu_y = \mu_{y_\circ}$ .

We show that  $\mu_x \circ \mu_y = \mu_{x_\circ} \circ \mu_{y_\circ}$ , i.e.  $\mu_{xy} = \mu_{x_\circ y_\circ}$ . Thus by (3.1)

$$\begin{aligned}
 \mu_{xy}(g) &= \mu(g(xy)^{-1}) = \mu(g \cdot x^{-1}y^{-1}) \quad \forall g \in G; \text{ and} \\
 \mu_{x_\circ y_\circ}(g) &= \mu(g(x_\circ y_\circ)^{-1}) = \mu(g \cdot x_\circ^{-1}y_\circ^{-1}) \quad \forall g \in G.
 \end{aligned}$$

Now  $\forall x, y \in G$ ,

$$\begin{aligned}
\mu(g \cdot x^{-1}y^{-1}) &= \mu[e(g \cdot x^{-1}y^{-1})] \\
&= \mu[((x_{\circ}y_{\circ})^{-1}(x_{\circ}y_{\circ}))(g \cdot x^{-1}y^{-1})] \\
&= \mu[((x_{\circ}^{-1}y_{\circ}^{-1})(x_{\circ}y_{\circ}))(g \cdot x^{-1}y^{-1})] \\
&= \mu[g(((x_{\circ}^{-1}y_{\circ}^{-1})(x_{\circ}y_{\circ}))(x^{-1}y^{-1}))] \text{ [in } G; a(bc) = b(ac) \text{ [8]]} \\
&= \mu[g(((x^{-1}y^{-1})(x_{\circ}y_{\circ}))(x_{\circ}^{-1}y_{\circ}^{-1}))] \text{ [using left invertive law]} \\
&= \mu[((x^{-1}y^{-1})(x_{\circ}y_{\circ}))(g(x_{\circ}^{-1}y_{\circ}^{-1}))] \text{ [in } G; a(bc) = b(ac) \text{ [8]]} \\
&\geq \mu((x^{-1}y^{-1})(x_{\circ}y_{\circ})) \wedge \mu(g(x_{\circ}^{-1}y_{\circ}^{-1}))
\end{aligned}$$

$$(3.2) \Rightarrow \mu(g \cdot x^{-1}y^{-1}) \geq \mu((x^{-1}y^{-1})(x_{\circ}y_{\circ})) \wedge \mu(g(x_{\circ}^{-1}y_{\circ}^{-1}))$$

Now we show that  $\mu((x^{-1}y^{-1})(x_{\circ}y_{\circ})) = \mu(e)$  in (3.2). Let  $\mu_x = \mu_{x_{\circ}} \Rightarrow \mu_x(g) = \mu_{x_{\circ}}(g) \forall g \in G$ ,

$$(3.3) \Rightarrow \mu(gx^{-1}) = \mu(gx_{\circ}^{-1}). \quad [\text{by (3.1)}]$$

Similarly, since  $\mu_y = \mu_{y_{\circ}} \Rightarrow \mu_y(g) = \mu_{y_{\circ}}(g) \forall g \in G$ ,

$$(3.4) \Rightarrow \mu(gy^{-1}) = \mu(gy_{\circ}^{-1}). \quad [\text{by (3.1)}]$$

Now,

$$\begin{aligned}
\mu((x^{-1}y^{-1})(x_{\circ}y_{\circ})) &= \mu((x_{\circ}y_{\circ} \cdot y^{-1})x^{-1}) \text{ [using left invertive law]} \\
&= \mu((x_{\circ}y_{\circ} \cdot y^{-1})x_{\circ}^{-1}) \\
&\quad [\text{substituting } g \text{ by } (x_{\circ}y_{\circ} \cdot y^{-1}) \text{ in (3.3)}] \\
&= \mu(x_{\circ}(y_{\circ}y^{-1} \cdot x_{\circ}^{-1})) \text{ [in } G; (ab \cdot c)d = a(bc \cdot d) \text{ [8]]} \\
&= \mu((y_{\circ}y^{-1})(x_{\circ}x_{\circ}^{-1})) \text{ [in } G; a(bc) = b(ac) \text{ [8]]} \\
&= \mu(y_{\circ}y^{-1} \cdot e) = \mu(ey^{-1} \cdot y_{\circ}) \text{ [using left invertive law]} \\
&= \mu(y^{-1}y_{\circ}) \\
&= \mu(y_{\circ}y^{-1}) \quad [\text{by Proposition 1}] \\
&= \mu(y_{\circ}y_{\circ}^{-1}) \quad [\text{substituting } g \text{ by } y_{\circ} \text{ in (3.4)}] \\
&= \mu(e).
\end{aligned}$$

Therefore, (3.2) implies that  $\mu(g \cdot x^{-1}y^{-1}) \geq \mu(g \cdot x_{\circ}^{-1}y_{\circ}^{-1})$ . Similarly one can prove that  $\mu(g \cdot x_{\circ}^{-1}y_{\circ}^{-1}) \geq \mu(g \cdot x^{-1}y^{-1})$ . Consequently,

$$\begin{aligned}
\mu(g \cdot x^{-1}y^{-1}) &= \mu(g \cdot x_{\circ}^{-1}y_{\circ}^{-1}) \\
\Rightarrow \mu(g \cdot (xy)^{-1}) &= \mu(g \cdot (x_{\circ}y_{\circ})^{-1}) \\
\Rightarrow \mu_{xy}(g) &= \mu_{x_{\circ}y_{\circ}}(g) \forall g \in G \\
\Rightarrow \mu_{xy} &= \mu_{x_{\circ}y_{\circ}}.
\end{aligned}$$

Hence the product of cosets is well-defined. Now we show that  $G/\mu$  form an AG-group under the operation  $\circ$ .

$G/\mu$  is closed under the operation  $\circ$ . Also  $G/\mu$  satisfies left invertive law under  $\circ$ ; since  $(\mu_x \circ \mu_y) \circ \mu_z = \mu_{xy} \circ \mu_z = \mu_{xy \cdot z} = \mu_{zy \cdot x} = \mu_{zy} \circ \mu_x = (\mu_z \circ \mu_y) \circ \mu_x \quad \forall x, y, z \in G$ . Now for any  $x \in G$ ,  $(\mu_e \circ \mu_x)(g) = (\mu_{ex})(g) = (\mu_x)(g) \Rightarrow (\mu_e \circ \mu_x) = \mu_x \quad \forall g \in G$ , but  $(\mu_x \circ \mu_e)(g) = (\mu_{xe})(g) \neq (\mu_x)(g) \Rightarrow (\mu_x \circ \mu_e) \neq \mu_x \quad \forall g \in G$ . This implies that  $\mu_e$  is the left identity of  $G/\mu$ . As an AG-group  $G$  is non associative therefore,  $(\mu_x \circ \mu_y) \circ \mu_z \neq \mu_x \circ (\mu_y \circ \mu_z)$ . Finally,  $\forall x \in G$ , once  $(\mu_x \circ \mu_{x^{-1}})(g) = (\mu_{xx^{-1}})(g) = (\mu_e)(g) \Rightarrow \mu_x \circ \mu_{x^{-1}} = \mu_e \quad \forall g \in G$ , and  $(\mu_{x^{-1}} \circ \mu_x)(g) = (\mu_{x^{-1}x})(g) = (\mu_e)(g) \Rightarrow \mu_{x^{-1}} \circ \mu_x = \mu_e \quad \forall g \in G$  the inverse of each  $\mu_x$  exists and is  $\mu_{x^{-1}}$ . Hence it follows that  $G/\mu$  is an AG-subgroup. ■

**Remark 1.** The AG-group  $G/\mu$  defined in Theorem 2 is called quotient AG-group of  $G$  relative to the normal fuzzy AG-subgroup  $\mu$ .

**Theorem 3.** Let  $\nu \in F(G)$  and  $H$  be any AG-subgroup of  $G$ . Define  $\xi \in FP(G/H)$  as follows:

$$\xi(Hx) = \bigvee \{ \nu(z) : z \in Hx \} \quad \forall x \in G.$$

Then  $\xi \in F(G/H)$ .

*Proof.* Since  $\forall x, y \in G$ ,

$$\begin{aligned} \xi(HxHy) = \xi(H(xy)) &= \bigvee \{ \nu(z) : z \in H(xy) \} \\ &= \bigvee \{ \nu(uv) : u \in Hx, v \in Hy \} \\ &\geq \bigvee \{ \nu(u) \wedge \nu(v) : u \in Hx, v \in Hy \} \\ &= (\bigvee \{ \nu(u) : u \in Hx \}) \wedge (\bigvee \{ \nu(v) : v \in Hy \}) \\ &= \xi(Hx) \wedge \xi(Hy) \end{aligned}$$

and  $\forall x \in G$ ,

$$\begin{aligned} \xi(Hx)^{-1} = \xi(Hx^{-1}) &= \bigvee \{ \nu(z) : z \in Hx^{-1} \} \\ &= \bigvee \{ \nu(w^{-1}) : w^{-1} \in Hx^{-1} \} \\ &\geq \bigvee \{ \nu(w) : w \in Hx \} \\ &= \xi(Hx). \end{aligned}$$

Hence  $\xi \in F(G/H)$ . ■

**Remark 2.** The fuzzy AG-subgroup defined in Theorem 3 is called Quotient fuzzy AG-subgroup or factor fuzzy AG-subgroup of  $G$ , and is denoted by  $\nu/H$ .

**Theorem 4.** Let  $\mu \in \mathcal{F}(G)$ . Then  $\mu_x = \mu_y \Leftrightarrow \mu_x = \mu_y \forall x, y \in G$ .

*Proof.* Let  $\mu_x = \mu_y$ , then

$$\begin{aligned} \mu_x(g) &= \mu_y(g) \forall g \in G \\ (3.5) \quad \Rightarrow \mu(gx^{-1}) &= \mu(gy^{-1}) \forall g \in G \quad [\text{using (3.1)}] \\ \text{put } g = y, \text{ in (3.5) we get } \mu(yx^{-1}) &= \mu(e) \Rightarrow yx^{-1} \in \mu_*. \quad [\text{by (2.1)}] \\ \Rightarrow (yx^{-1})x &\in \mu_* \Rightarrow (xx^{-1})y \in \mu_* \quad (\text{using left invertive law}) \Rightarrow y \in \mu_*, \\ \text{but } y &\in \mu_*, \text{ therefore, } \mu_y \subseteq \mu_x. \end{aligned}$$

Again put  $g = x$ , in (3.5) we get  $\mu(xx^{-1}) = \mu(xy^{-1}) \Rightarrow \mu(xy^{-1}) = \mu(e) \Rightarrow xy^{-1} \in \mu_*$ . [by (2.1)]  $\Rightarrow (xy^{-1})y \in \mu_* \Rightarrow x \in \mu_*$ , but  $x \in \mu_*$ . This implies that,  $\mu_x \subseteq \mu_y$ . Thus  $\mu_x \subseteq \mu_y \subseteq \mu_x$ . Hence  $\mu_x = \mu_y$ .

Conversely; let  $\mu_x = \mu_y \Rightarrow \mu_x \circ \mu_y = \mu_x \circ \mu_y \Rightarrow \mu_x = \mu_y \Rightarrow \mu_x = \mu_y$ . Now for any  $x, y \in G$  it follows that

$$\begin{aligned} \mu(gx^{-1}) &= \mu(g((y^{-1}y)x^{-1})) \\ &= \mu(g((x^{-1}y)y^{-1})) \quad [\text{using left invertive law}] \\ &= \mu((x^{-1}y)(gy^{-1})) \quad [\text{in } G; a(bc) = b(ac) [8]] \\ &\geq \mu(x^{-1}y) \wedge \mu(gy^{-1}) \quad [\mu \in \mathcal{F}(G)] \\ &= \mu((xy^{-1})^{-1}) \wedge \mu(gy^{-1}) \\ &= \mu(xy^{-1}) \wedge \mu(gy^{-1}) \quad [\text{by Lemma 1-(ii)}] \\ &= \mu(e) \wedge \mu(gy^{-1}) \quad [\text{by (2.1); as } xy^{-1} \in \mu_*] \\ &= \mu(gy^{-1}) \quad [\text{by Lemma 1-(i)}] \end{aligned}$$

This implies that  $\mu(gx^{-1}) \geq \mu(gy^{-1})$ .

By similar arrangements we can show that,  $\mu(gy^{-1}) \geq \mu(gx^{-1})$ . Consequently  $\mu(gx^{-1}) = \mu(gy^{-1}) \Rightarrow \mu_x(g) = \mu_y(g) \forall g \in G$  by (3.1). Hence  $\mu_x = \mu_y$ . ■



**Theorem 5.** *Let  $\mu \in NF(G)$  and  $\mu_x = \mu_y$ , then  $\mu(x) = \mu(y) \forall x, y \in G$ .*

*Proof.* Let  $x, y \in G$ , then  $\mu_x = \mu_y \Leftrightarrow \mu_x = \mu_y \Rightarrow \mu_x \circ \mu_y = \mu_x \circ \mu_y \Rightarrow \mu_x = \mu_y \Rightarrow \mu_x = \mu_y \Rightarrow xy^{-1} \in \mu_*$ ; (using Theorem 4 and the definition of fuzzy cosets). Therefore,

$$\begin{aligned}
 \mu(y) = \mu(y^{-1}) &= \mu(x^{-1}y^{-1} \cdot (x^{-1})^{-1}) \quad [\mu \in NF(G)] \\
 &= \mu(x^{-1}y^{-1} \cdot x) \\
 &= \mu(xy^{-1} \cdot x^{-1}) \quad [\text{using left invertive law}] \\
 &\geq \mu(xy^{-1}) \wedge \mu(x^{-1}) \\
 &= \mu(e) \wedge \mu(x) \quad [\text{by (2.1), as } xy^{-1} \in \mu_*] \\
 &= \mu(x) \quad [\text{by Lemma 1-(i)}] \\
 \Rightarrow \mu(y) &\geq \mu(x).
 \end{aligned}$$

Similarly, we can show that  $\mu(x) \geq \mu(y)$ . This implies that  $\mu(x) \geq \mu(y) \geq \mu(x)$ . Hence  $\mu(x) = \mu(y)$ . ■

**Proposition 2.** *Let  $\mu \in NF(G)$ . Then  $\mu_x(xg) = \mu_x(gx) = \mu(g) \forall g \in G$ .*

*Proof.* Using definition of cosets of fuzzy AG-subgroup, it follows that for  $g \in G$ ;  $\mu_x(xg) = \mu(xg \cdot x^{-1}) = \mu(g)$ . And  $\mu_x(gx) = \mu(gx \cdot x^{-1}) = \mu(x^{-1}x \cdot g) = \mu(eg) = \mu(g)$ . Hence  $\mu_x(xg) = \mu_x(gx) = \mu(g) \forall g \in G$ . ■

**Theorem 6.** *Let  $\mu \in NF(G)$ . Then the following assertions hold:*

- (i)  $G/\mu \cong G/\mu_*$ ;
- (ii) If  $\nu \in FP(G/\mu)$ ; defined by  $\nu(\mu_x) = \mu(x) \forall x \in G$ . Then  $\nu \in NF(G/\mu)$ .

*Proof.* As both  $G/\mu$  and  $G/\mu_*$  are AG-groups by Theorem 2 and  $\phi : G/\mu \rightarrow G/\mu_*$  given by  $\phi(\mu_x) = \mu_* \underset{x}{\forall} x \in G$  is an isomorphism by Theorem 4 and the fact that  $\mu_x \circ \mu_y = \mu_{xy}$  and  $\mu_* \underset{x}{\circ} \mu_* \underset{y}{=} \mu_* \underset{xy}{\cdot}$ .

(ii) Let  $\nu \in FP(G/\mu)$ , be defined by  $\nu(\mu_x) = \mu(x) \forall x \in G$ . We show that  $\nu \in NF(G/\mu)$ . Since

$$\begin{aligned} \nu(\mu_x \circ \mu_y) &= \nu(\mu_{xy}) \\ &= \mu(xy) && [\text{by definition of } \nu] \\ &\geq \mu(x) \wedge \mu(y) && [\mu \in NF(G)] \\ &= \nu(\mu_x) \wedge \nu(\mu_y) && [\text{by definition of } \nu] \end{aligned}$$

$\forall x, y \in G$ , and

$$\nu((\mu_x)^{-1}) = \nu(\mu_{x^{-1}}) = \mu(x^{-1}) \geq \mu(x) = \nu(\mu_x)$$

$\forall x \in G$ . Hence  $\nu \in F(G/\mu)$ . Further, since

$$\begin{aligned} \nu((\mu_x \circ \mu_y) \circ (\mu_x)^{-1}) &= \nu(\mu_{xy} \circ \mu_{x^{-1}}) \\ &= \nu(\mu_{xy \cdot x^{-1}}) \\ &= \mu(xy \cdot x^{-1}) && [\text{by definition of } \nu] \\ &= \mu(y) && [\mu \in NF(G)] \\ &= \nu(\mu_y). && [\text{by definition of } \nu] \end{aligned}$$

$\forall x, y \in G$ . Hence  $\nu \in NF(G/\mu)$ . ■

**Theorem 7.** Let  $\mu \in NF(G)$ . Define a mapping  $\theta : G \rightarrow G/\mu$  as follows:

$$(3.6) \quad \theta(x) = \mu_x \forall x \in G.$$

Then  $\theta$  is homomorphism with kernel  $\mu_*$ .

*Proof.* Since

$$\theta(xy) = \mu_{xy} = \mu_x \circ \mu_y = \theta(x)\theta(y) \quad \forall x, y \in G.$$

Hence  $\theta$  is homomorphism. Further, the kernel of  $\theta$  consists of all  $x \in G$  for which  $\mu_x = \mu_e \Leftrightarrow \mu(x) = \mu(e)$ , (by Theorem 5)  $\Leftrightarrow x \in \mu_*$ . Thus  $Ker\theta = \mu_*$ . ■

**Theorem 8.** Let  $\mu \in NF(G)$ , and  $G/\mu$  is an AG-group. Then each  $\zeta \in NF(G/\mu)$  corresponds in a natural way to  $\nu \in NF(G)$ .

*Proof.* Let  $\zeta \in NF(G/\mu)$ . Define a mapping  $\nu : G \rightarrow [0, 1]$  as follows:

$$\nu(x) = \zeta(\mu_x) \quad \forall x \in G.$$

First we show that  $\nu \in F(G)$ . Since  $\forall x, y \in G$ ,

$$\begin{aligned} \nu(xy) &= \zeta(\mu_{xy}) \\ &= \zeta(\mu_x \circ \mu_y) \\ &\geq \zeta(\mu_x) \wedge \zeta(\mu_y) \quad [\zeta \in NF(G/\mu)] \\ &= \nu(x) \wedge \nu(y) \end{aligned}$$

and  $\forall x \in G$ ,

$$\nu(x^{-1}) = \zeta(\mu_{x^{-1}}) = \zeta(\mu_x)^{-1} \geq \zeta(\mu_x) = \nu(x)$$

Thus  $\nu \in F(G)$ .

Further, since  $\forall x, y \in G$ ,

$$\begin{aligned} \nu(xy \cdot x^{-1}) &= \zeta(\mu_{xy \cdot x^{-1}}) \\ &= \zeta(\mu_y) \quad [\mu \in NF(G)] \\ &= \nu(y). \end{aligned}$$

Hence  $\nu \in NF(G)$ . ■

In the following we introduce fuzzy Lagrange's Theorem for AG-group of finite order. We start with the following definition.

**Definition 4.** Let  $G$  be a finite AG-group,  $\mu \in F(G)$  and  $G/\mu$  is an AG-group. Then the cardinality of  $G/\mu$  is called the index of fuzzy AG-subgroup of  $\mu$  in  $G$  written as  $[G : \mu]$ .

**Theorem 9.** (Fuzzy Lagrange's Theorem for AG-subgroup). Let  $G$  be a finite AG-group,  $\mu \in F(G)$ . Then the index of fuzzy AG-subgroup of  $\mu$  divides the order of  $G$ .

*Proof.* It follows from Theorem 7, that there is homomorphism  $\theta$  from  $G$  into  $G/\mu$ , the set of all fuzzy cosets of  $\mu$ , defined in (3.6). Let  $H$  be an AG-subgroup of  $G$  defined by  $H = \{h \in G : \mu_h = \mu_e\}$ . Let  $h \in H$ , then  $\mu_h = \mu_e \Leftrightarrow \mu_h = \mu_e$  using Theorem 4. Therefore,  $H = \{h \in G : \mu_h = \mu_e\}$ . Now decomposing  $G$  as a disjoint union of the cosets of  $G$  with respect to  $H$  i.e.

$$(3.7) \quad G = (H = Hx_1) \cup Hx_2 \cup \cdots \cup Hx_k,$$

where  $x_1 \in H$  and  $x_i \in G$ ;  $1 \leq i \leq k$ . Now, we show that corresponding to each coset  $Hx_i$ ;  $1 \leq i \leq k$ , given in (3.7) there is a fuzzy coset belonging to  $G/\mu$ , and further this correspondence is one-one. To see this, consider any coset  $Hx_i$  for any  $h \in H$ , we have that;  $\theta(hx_i) = \mu_{hx_i} = \mu_h \circ \mu_{x_i} = \mu_e \cdot \mu_{x_i} = \mu_{ex_i} = \mu_{x_i}$ . Thus  $\theta$  maps each element of  $Hx_i$  into the fuzzy cosets  $\mu_{x_i}$ .

Now we show that  $\theta$  is well-defined. Let  $Hx_i = Hx_j$ , for each  $i, j : 1 \leq i \leq k$  and  $1 \leq j \leq k$ . Then

$$\begin{aligned}
x_j^{-1}x_i &\in H && [\text{cosets in AG-groups}] \\
\Rightarrow \mu_{x_j^{-1}x_i} &= \mu_e \\
\Rightarrow \mu_{(x_j^{-1}x_i)} &= \mu_e && [\text{by Theorem 4}] \\
\Rightarrow \mu_{(x_j^{-1}x_i)} \circ \mu_{x_i^{-1}} &= \mu_e \circ \mu_{x_i^{-1}} \\
\Rightarrow \mu_{(x_j^{-1}x_i)x_i^{-1}} &= \mu_{ex_i^{-1}} \\
\Rightarrow \mu_{(x_i^{-1}x_i)x_j^{-1}} &= \mu_{ex_i^{-1}} && [\text{using left invertive law}] \\
\Rightarrow \mu_{x_i^{-1}} &= \mu_{x_j^{-1}} \\
\Rightarrow \mu_{x_i} &= \mu_{x_j} \\
\Rightarrow \mu_{x_i} &= \mu_{x_j} && [\text{by Theorem 4}] \\
\Rightarrow \theta(Hx_i) &= \theta(Hx_j).
\end{aligned}$$

Thus  $\theta$  is well-defined.

Further, we show that  $\theta$  is one-one; for each  $i, j$  where  $1 \leq i \leq k$  and  $1 \leq j \leq k$ ; assume that

$$\begin{aligned}
\theta(Hx_i) &= \theta(Hx_j) \\
\Rightarrow \mu_{x_i} &= \mu_{x_j} \\
\Rightarrow \mu_{x_i} &= \mu_{x_j} && [\text{Theorem 4}] \\
\Rightarrow \mu_{x_i^{-1}} &= \mu_{x_j^{-1}} && [\text{cosets in AG-groups}] \\
\Rightarrow \mu_e \circ \mu_{x_i^{-1}} &= \mu_e \circ \mu_{x_j^{-1}}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \mu_{*e x_i^{-1}} = \mu_{*e x_j^{-1}} \\
&\Rightarrow \mu_{*e x_i^{-1}} = \mu_{*(x_i^{-1} x_i \cdot x_j^{-1})} \\
&\Rightarrow \mu_{*e x_i^{-1}} = \mu_{*(x_j^{-1} x_i \cdot x_i^{-1})} \\
&\Rightarrow \mu_{*(x_j^{-1} x_i)} \circ \mu_{*x_i^{-1}} = \mu_{*e} \circ \mu_{*x_i^{-1}} \\
&\Rightarrow \mu_{*(x_j^{-1} x_i)} = \mu_{*e} \quad [\text{by Theorem 4}] \\
&\Rightarrow x_j^{-1} x_i \in H
\end{aligned}$$

$$\Leftrightarrow H x_i = H x_j \quad [\text{for each } i \text{ and } j \text{ where } 1 \leq i \leq k \text{ and } 1 \leq j \leq k.]$$

From above discussion it is now clear that the number of distinct cosets of  $H$  (index) in  $G$  equals the number of fuzzy cosets of  $\mu$ , which is a divisor of the order of  $G$ . Hence we conclude that the index of  $\mu$  also divides the order of  $G$ . ■

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